Equivalence Checking of Quantum Circuits

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DQC Scientific Quantum Conference

DEPARTMENT OF COMPUTER SCIENCE

This talk is based on joint work[1]

Christian Bøgh Larsen

Simon Brun Olsen

Kim Guldstrand Larsen

INDEPENDENT RESEARCH FUND DENMARK

VILLUM FONDEN

DIREC
Digital Research Centre Denmark

[1] C. B. Larsen, S. B. Olsen, K. G. Larsen, and C. Schilling. "Contraction heuristics for tensor decision diagrams". *Entropy* (2024).

Overview

Equivalence checking of quantum circuits

Tensor networks and tensor decision diagrams

Equivalence checking based on tensor decision diagrams

Empirical evaluation

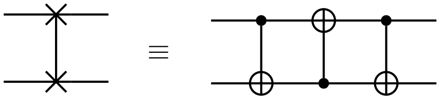
Conclusion and future work

Quantum circuit compilation

- When designing quantum algorithms, it is useful to have many types of operations available
- Real quantum computers only support a few types of operations
- A **compiler** translates a high-level circuit with many gate types to a low-level circuit with few gate types
- Important that the compiled circuits are **equivalent** (i.e., compute the same output for the same input)

How to implement a SWAP gate?

- The SWAP gate is **equivalent** to three CNOT gates



How to implement a SWAP gate?

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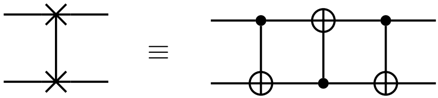

- We can easily prove this by comparing the matrices

$$\begin{bmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{bmatrix}=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&0&0&1\\0&0&1&0\\0&1&0&0\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}$$

Algorithm to check equivalence of quantum circuits

- Given: Two circuits C_1, C_2
- Question: Are the circuits equivalent ($C_1 \equiv C_2$)?

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  1. Compute matrix representations U_1, U_2
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$$U_1 \stackrel{?}{=} e^{i\theta} \cdot U_2$$

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- Problem: Matrices are **exponentially large**

  For n qubits $\rightsquigarrow 2^n \times 2^n$

Equivalence checking is hard

- Checking exact equivalence is NQP-complete[1]
- Checking approximate equivalence is QMA-complete[2,3]
- Problems in these complexity classes are widely believed to require exponential computations in the worst case

[1] Y. Tanaka. "Exact non-identity check is NQP-complete". *Int. J. Quantum Inf.* (2010).
[2] D. Janzing, P. Wocjan, and T. Beth. ""Non-identity-check" is QMA-complete". *Int. J. Quantum Inf.* (2005).
[3] Z. Ji and X. Wu. *Non-identity check remains QMA-complete for short circuits*. 2009. arXiv: 0906.5416.

Approaches to equivalence checking

- ZX-calculus[1]
- Encoding with decision diagrams[2] ← relevant later
- Tensor network contraction[3] ← relevant later
- Simulation-based approach for the Clifford group[4]
- Weighted model counting[5]

[1] T. Peham, L. Burgholzer, and R. Wille. "Equivalence checking of quantum circuits with the ZX-calculus". *IEEE J. Emerg. Sel. Topics Circuits Syst.* (2022).

[2] L. Burgholzer and R. Wille. "Advanced equivalence checking for quantum circuits". *IEEE Trans. Comput. Aided Des. Integr. Circuits Syst.* (2021).

[3] R. Orús. "Tensor networks for complex quantum systems". *Nature Reviews Physics* (2019).

[4] D. Thanos, T. Coopmans, and A. Laarman. "Fast equivalence checking of quantum circuits of Clifford gates". *ATVA*. 2023.

[5] J. Mei, T. Coopmans, M. M. Bonsangue, and A. Laarman. "Equivalence checking of quantum circuits by model counting". *IJCAR*. 2024.

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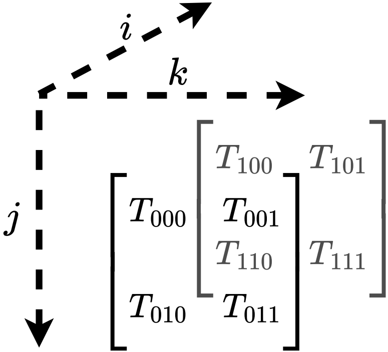
Tensors

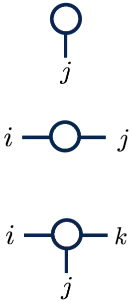
- Generalization of vectors / matrices to higher dimensions



Tensors

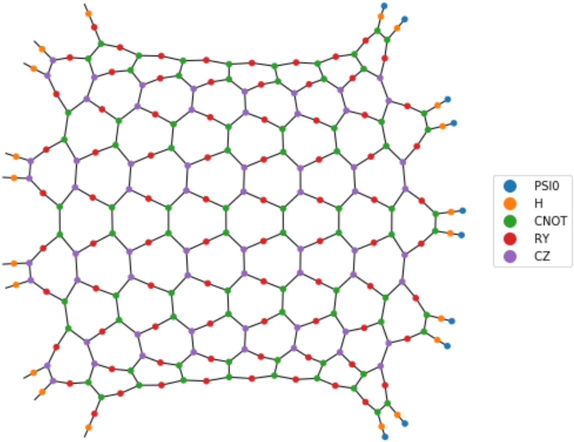
- Generalization of vectors / matrices to higher dimensions
- High-level graphical representation as a node with edges

vector	v_j
matrix	M_{ij}
3-index tensor	T_{ijk}



Tensor networks

- Tensors can be arranged in a graph

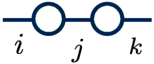


Tensor networks

- Tensors can be arranged in a graph
- Nodes with shared edges can be **contracted** (= merged)

  Corresponds to matrix-vector and matrix-matrix multiplication for special cases

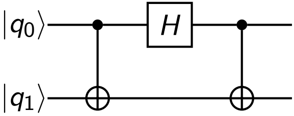
$$= \sum_j A_{ij} v_j$$

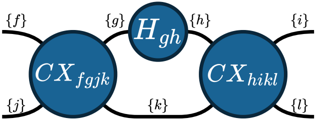


$$= \sum_j A_{ij} B_{jk} = AB$$

Example

1. Initial tensor network has one tensor for each gate
   Three choices for contraction $(\{g\}, \{h\}, \{k\})$





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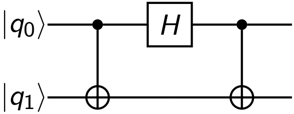
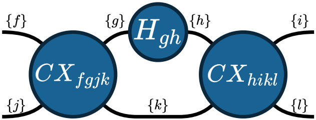
Example

1. Initial tensor network has one tensor for each gate
   Three choices for contraction $(\{g\}, \{h\}, \{k\})$
2. Contraction of CX_{fgjk} and H_{gh} via $\{g\}$





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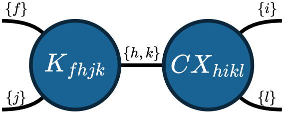
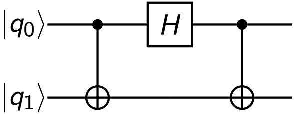
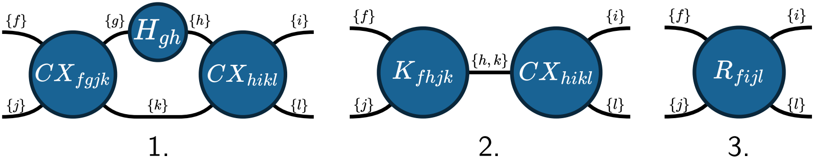


2.

Example

1. Initial tensor network has one tensor for each gate
   Three choices for contraction $(\{g\}, \{h\}, \{k\})$
2. Contraction of CX_{fgjk} and H_{gh} via $\{g\}$
3. Contraction of remaining two tensors





Application: Quantum simulation on classical computer[2]

- Contract tensors in smart orders
- Different contraction heuristics to minimize floating-point operations, size, etc.[1]

[1] J. Gray and S. Kourtis. "Hyper-optimized tensor network contraction". *Quantum* (2021).

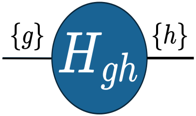
[2] I. L. Markov and Y. Shi. "Simulating quantum computation by contracting tensor networks". *SIAM J. Comput.* (2008).

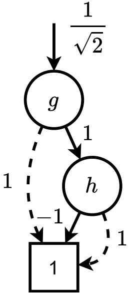
Tensor decision diagrams (TDDs)

- Alternative, unique representation of a tensor
- Informal introduction by example

Example: Hadamard gate with matrix, tensor, and TDD

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$





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Alternative "reverse scheme" for equivalence[1]

$$C_1 \equiv C_2 \overset{\text{def}}{\iff} \exists\theta\colon U_1 = e^{i\theta} \cdot U_2$$

$$\iff \exists\theta\colon U_1 \cdot U_2^{\dagger} = e^{i\theta} \cdot I \overset{\text{def}}{\iff} C_1 C_2^{-1} \equiv C_I$$

- C_2^{-1} is the inverted C_2 (reversed and each gate inverted)
- Allows to combine both circuits

[1] G. F. Viamontes, I. L. Markov, and J. P. Hayes. "Checking equivalence of quantum circuits and states". *ICCAD*. 2007.

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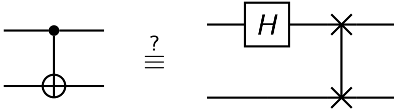
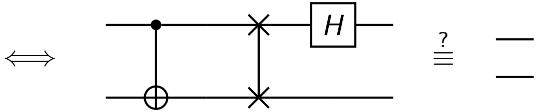




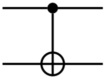
- (Coincidentally, the swap and Hadamard gates are self-inverse)

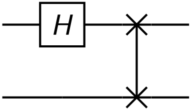
[1]G. F. Viamontes, I. L. Markov, and J. P. Hayes. "Checking equivalence of quantum circuits and states". *ICCAD*. 2007.

New algorithm for equivalence checking

Algorithm combines reverse scheme, tensor networks, and TDDs

Given: Quantum circuits C_1, C_2





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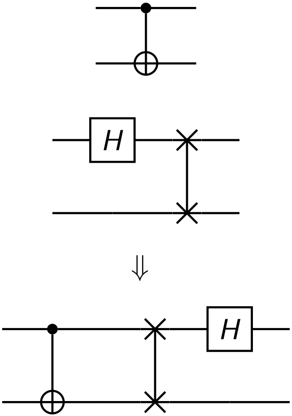

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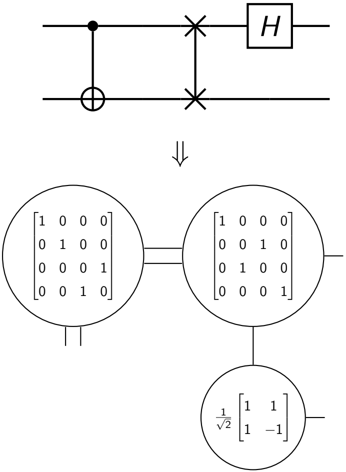
Given: Quantum circuits C_1, C_2

1. Construct circuit $C_1 C_2^{-1}$
2. Convert $C_1 C_2^{-1}$ to tensor network



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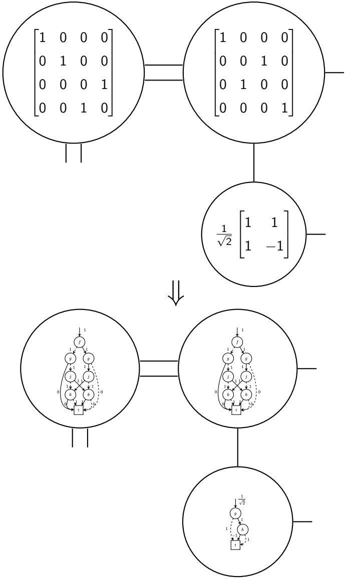
Given: Quantum circuits C_1, C_2

1. Construct circuit $C_1 C_2^{-1}$
2. Convert $C_1 C_2^{-1}$ to tensor network
3. Convert all tensors to TDDs



(TDDs on the right are only exemplary)

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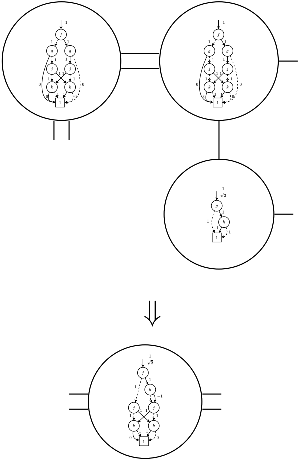

Given: Quantum circuits C_1, C_2

1. Construct circuit $C_1 C_2^{-1}$
2. Convert $C_1 C_2^{-1}$ to tensor network
3. Convert all tensors to TDDs
4. Contract TDD network

(TDDs on the right are only exemplary)

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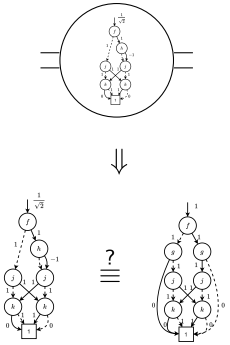

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5. Compare TDD to identity TDD

(TDDs on the right are only exemplary)

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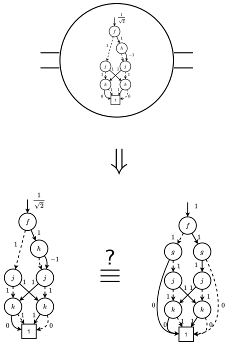
3. Convert all tensors to TDDs
4. **Contract TDD network** $\leftarrow$ how?
5. Compare TDD to identity TDD



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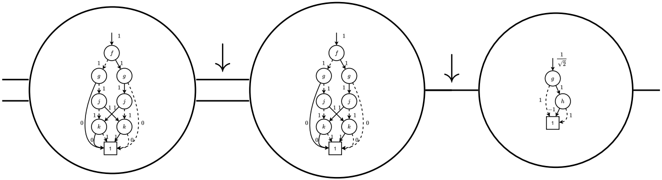
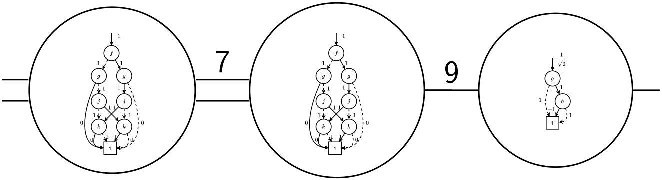
Lookahead heuristic for TDD contraction

- Greedy algorithm (finds a local optimum)
- In each contraction step:
  1. Evaluate all possible contractions



Lookahead heuristic for TDD contraction

- Greedy algorithm (finds a local optimum)
- In each contraction step:
  1. Evaluate all possible contractions
  2. Measure size of resulting TDDs



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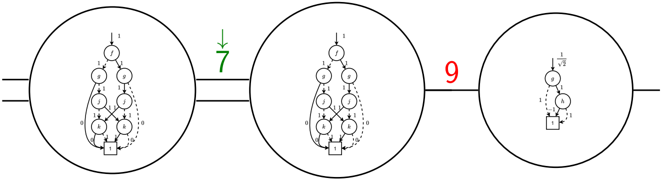

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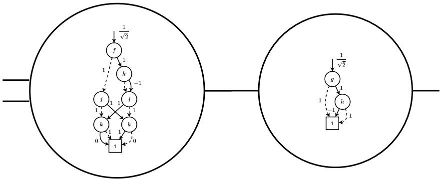

Lookahead heuristic for TDD contraction

- Greedy algorithm (finds a local optimum)
- In each contraction step:
  1. Evaluate all possible contractions
  2. Measure size of resulting TDDs
  3. Execute a contraction with smallest output
- Why should this scale?
  - Network is sparsely connected
  - Initial contractions (with many tensors) are cheap
  - Results can be stored for later iterations
- Still, step 1. is quite expensive

Counting heuristic for TDD contraction

- Goal: Imitate lookahead heuristic without expensive step 1
- Empirical observation: Lookahead heuristic prefers to distribute the contractions over the tensor network

[1] We set one edge to "2" to get an interesting example

Counting heuristic for TDD contraction

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- In each contraction step:
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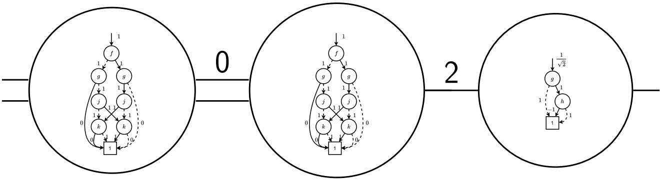

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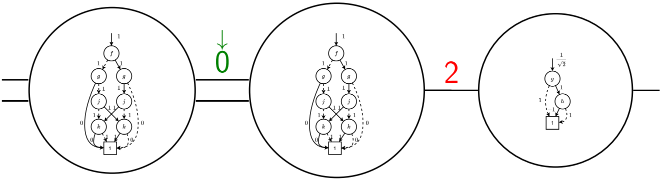

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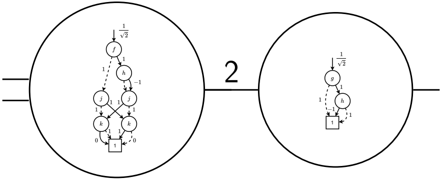

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Counting heuristic for TDD contraction

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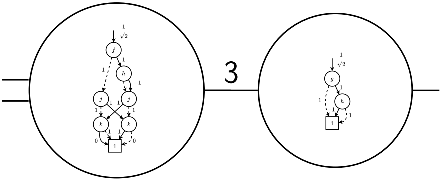
  2. Update usage statistics of neighboring edges



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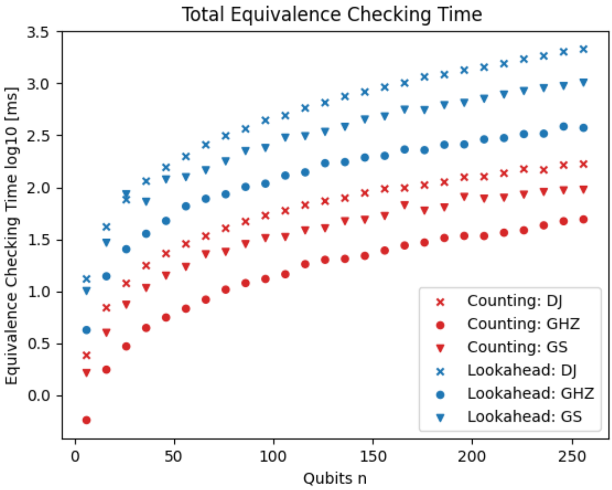
Quantum circuits in evaluation

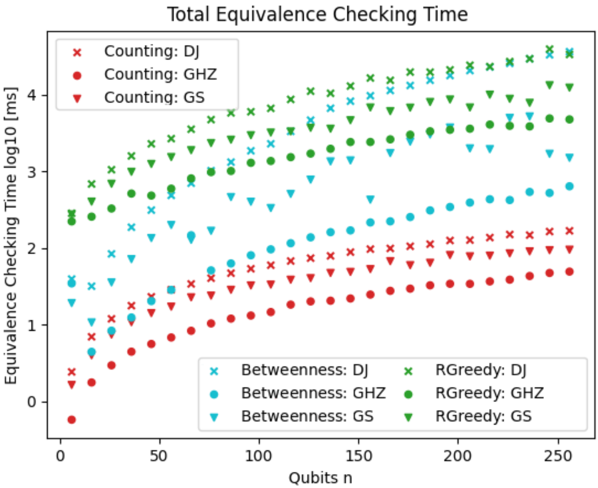
- Circuits from MQT Bench[1] with varying number of qubits at two compilation levels (level 1 and 3 (out of 4)) with significantly different gate sets and layouts
  - Deutsch-Jozsa algorithm (DJ)
  - Greenberger-Horne-Zeilinger state preparation (GHZ)
  - Graph state preparation (GS)

[1] N. Quetschlich, L. Burgholzer, and R. Wille. "MQT Bench: Benchmarking software and design automation tools for quantum computing". *Quantum* (2023).

Comparison of own heuristics

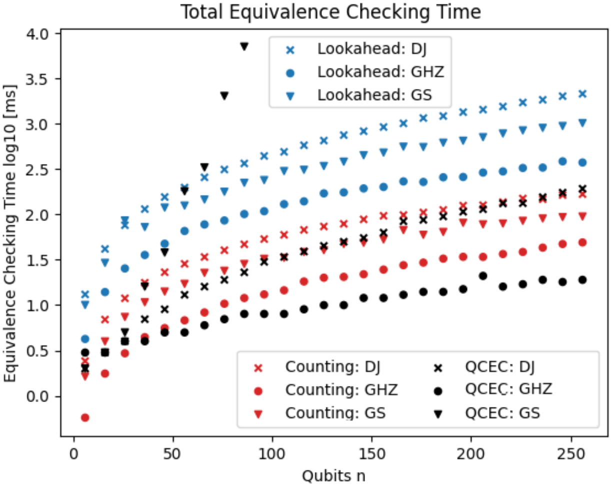


Comparison to cotengra[1]



[1] J. Gray and S. Kourtis. "Hyper-optimized tensor network contraction". *Quantum* (2021).

Comparison to QCEC[1]



[1] L. Burgholzer and R. Wille. "QCEC: A JKQ tool for quantum circuit equivalence checking". *Softw. Impacts* (2021).

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Conclusion

- Integration of "reverse scheme" and TDD networks
- Lookahead heuristic (greedy)
- Counting heuristic (cheap approximation)
- Evaluation:
  - Outperforms cotengra's heuristics
  - Often keeps up with QCEC

Future work & FMQC 2025

- Exploit parallelization (CPU, GPU)
- Find other heuristics
  - Generalize tensor network heuristics to TDD networks
  - Identify equivalent subcomponents (modularity)
  - Employ machine learning
- New PhD project since December 2024

DeiC

Workshop on Formal Methods in Quantum Computing

co-located with CONFEST 2025 in Aarhus, August 25

https://fmqc-workshop.github.io/2025/