Rigorous time evolution of p-boxes in non-linear ODEs

Ander Gray
Culham Science Centre, United Kingdom Atomic Energy Authority. E-mail: ander.gray@ukaea.uk

Marcelo Forets
Universidad de la República, Montevideo, Uruguay. E-mail: mforets@fing.edu.uy

Christian Schilling
Aalborg University, Aalborg, Denmark. E-mail: christianms@cs.aau.dk

Luis Benet
Instituto de Ciencias Físicas, Universidad Nacional Autónoma de México. E-mail: benet@icf.unam.mx

Scott Ferson
Institute for Risk and Uncertainty, University of Liverpool, UK. E-mail: scott.ferson@liverpool.ac.uk

We combine reachability analysis and probability bounds analysis, which allow for imprecisely known random variables (multivariate intervals or p-boxes) to be specified as the initial states of a dynamical system. In combination, the methods allow for the temporal evolution of p-boxes to be rigorously computed, and they give interval probabilities for *formal verification* problems, also called *failure probability* calculations in reliability analysis. The methodology places no constraints on the input probability distribution or p-box and can handle dependencies generally in the form of copulas.

Keywords: reachability analysis, probability bounds analysis, automatically verified computing, set-based methods, p-box, uncertainty propagation.

1. Reachability Analysis

Reachability analysis studies the rigorous time evolution of sets in non-linear dynamical systems. Usually reachability problems are presented as an interval initial value problem, where a bounded set of trajectories starting from an interval box is rigorously computed. Figure 1 shows an example of a two-dimensional set propagated through a non-linear dynamical system. The set-based dynamics are solved with Taylor models (Benet et al. 2019), which for specific time intervals $[t_i, t_{i+1}]$ rigorously represent the possible system states with a taylor series plus an interval remainder. The Taylor model reach sets are integrated through time using a verified Picard iteration, giving a rigorous outer approximation of the set of trajectories from the initial interval. For verification problems (reliability analysis) the method evaluates three possible probabilities: $\mathbb{P}_f = 0$ (guaranteed to be safe), $\mathbb{P}_f = 1$ (guaranteed to be unsafe), and the interval probability $\mathbb{P}_f = [0, 1]$ (unknown safety). The last of these is the case for the grey failure domain in Figure 1, and this imprecision is a drawback of the method.

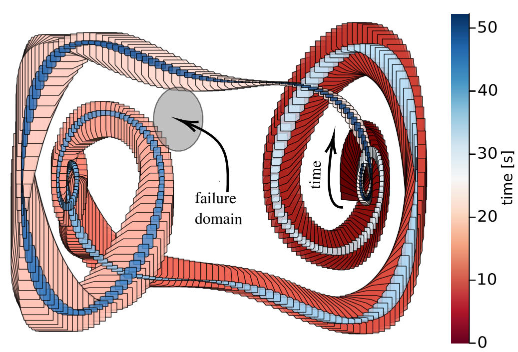

Fig. 1. Colours show time evolution of an interval ODE, and the grey set is a hypothetical elliptical failure domain which just touches the reach sets. Produced using ***ReachabilityAnalysis.jl*** (Bogomolov et al. 2019)

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2. Probability Bounds Analysis

We extend reachability analysis to allow for p-boxes to be specified as inputs, giving rigorous bounds on failure probabilities. Probability boxes (Gray et al. 2022) represent a set of probability distributions using interval bounds on cdfs and are a generalisation of both intervals and distributions. A multivariate p-box can be constructed with imprecise Sklar's theorem (Montes et al. 2015) using N marginal p-boxes and a single n-copula to capture stochastic dependence. A rigorous outer approximation (a belief function) of a p-box can be constructed using a finite set of intervals X (focal elements) and probability masses m. In the multivariate case the H-volume may be used to assign the masses to the focal elements using the copula. The focal elements may then be propagated using interval analysis $Y = f(X)$, with the masses conserved by each interval $m(X) = m(f(X))$. The output p-box can be reconstructed, and the interval failure probability computed on some domain U using the belief (lower) and plausibility (upper) measures

$$\underline{\mathbb{P}}(U) = \sum_{Y \subseteq U} m(Y),$$

$$\overline{\mathbb{P}}(U) = \sum_{Y \cap U \neq \emptyset} m(Y).$$

3. Imprecise Probabilistic Reachability

We first take the support of the multivariate p-box and perform a single reachability calculation, getting the Taylor model approximation using the entire input domain. Subsets of this input domain (focal elements) can be tightly propagated through the Taylor model using interval arithmetic. This allows us to perform the probability bounding calculation as a supplementary extra to a single reachability calculation if required, for example if the failure domain cannot be proven to be safe.

Setting the inputs to Figure 1 as the beta distributed p-boxes $X_1 \sim \text{beta}([2,3],[3,4])$ and $X_2 \sim \text{beta}([7,8],[2,3])$ defined by interval ranges for the traditional beta parameters, and with a correlation of $\rho = -0.8$ using a Gaussian copula, Figure 2 shows the time evolution of X_2 between 0 and 5 seconds. Further, it can be proven that

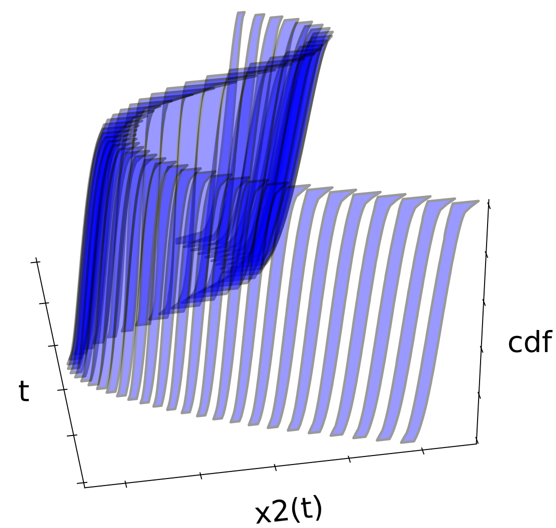

Fig. 2. Time evolution of a p-box for X_2 of the ODE from Figure 1 in $t = [0, 5]$s.

the failure probability is in the interval $\mathbb{P}_f = [0, 0.00367578]$, with the interval width coming from the imprecision of the input p-boxes, and the rigorous discretisation error of the p-box (only 100 focal elements per dimension) and outer approximation from the system dynamics.

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